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Noncommutative Chern-Simons soliton

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We have studied noncommutative extension of the relativistic Chern-Simons-Higgs model, in the first nontrivial order in θ, with only spatial noncommutativity. Both Lagrangian and Hamiltonian formulations of the problem have been discussed, with the focus being on the canonical and symmetric forms of the energy-momentum tensor. In the Hamiltonian scheme, constraint analysis and the induced Dirac brackets have been provided. The space-time translation generators and their actions on the fields are discussed in detail. The effects of noncommutativity on the soliton solutions have been analyzed thoroughly and we have come up with some interesting observations. Considering the *relative* strength of the noncommutative effects, we have shown that there is a universal character in the noncommutative correction to the magnetic field—it depends *only* on θ. On the other hand, in the cases of all other observables of physical interest, such as the potential profile, soliton mass, or the electric field, θ as well as τ (the latter comprised solely of commutative Chern-Simons-Higgs model parameters) appear on similar footings. Lastly, we have shown that noncommutativity imposes a further restriction on the form of the Higgs field so that the Bogomolny-Prasad-Sommerfeld equations are compatible with the full variational equations of motion.

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I. INTRODUCTION

The Chern-Simons electrodynamics has created a lot of interest in the past. Here the gauge-field dynamics is governed solely by the Chern-Simons term. The gauge theoretic part of this truncation can be perceived as the $\mu \rightarrow \infty$ limit of the topologically massive model [1],

$$L_{\rm top} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{\mu}{4}\epsilon^{\alpha\beta\gamma}F_{\alpha\beta}A_{\gamma}, \qquad (1)$$

$F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$ being the Abelian field tensor. Physically this is realizable in the context of large distance or low energy scales where the Chern-Simons term, with a lower number of derivatives, dominates over the higher derivative Maxwell term. The charged scalar field, minimally coupled to U(1) Chern-Simons gauge field, with a Higgs type of polynomial potential, gives rise to the celebrated Chern-Simons vortices [2,3]. For a particular potential profile, one gets a self-dual set of Bogomolny-Prasad-Sommerfeld (BPS) equations where, at the self-dual point, the solitons saturate the energy lower bound. These solitons have played significant roles in the context of anyonic quantum field theories [4].

After the relevance of noncommutative (NC) quantum field theories [5] was established in the context of string theory [6], soliton solutions in NC theory have generated a great amount of interest. Of the various types of NC solitons discussed so far [7–13], in the small θ regime (θ being the NC parameter), some NC solitons smoothly reach their corresponding commutative soliton limit [8,11–13], whereas in some cases [7,9,10], the $\theta \rightarrow 0$ limit is singular.

The importance of the Chern-Simons theory in commutative space-time [4] has led to investigations with the NC generalization of the Chern-Simons term [14]. Solitons in the NC Chern-Simons-Higgs system have been studied in the operatorial framework [10,11]. Interestingly, Bak *et. al.* [10] and Lozano *et. al.* in [11] both consider the NC Chern-Simons-Higgs system and obtain solitons pertaining to different branches, that is, solitons with singular or smooth commutative limit, respectively.[1]

Essentially there are two distinct ways of studying a NC field theory. (Indeed, they are expected to lead to a unique result.) The more common way is to reinterpret the fields as operators in the Hilbert space which carries a representation of the basic noncommutative geometry algebra. In fact *all* the works on NC solitons mentioned so far [7–12] follow this route. The other alternative is to construct the NC field theory from the commutative one by replacing ordinary products by Moyal (or $*$) products, the latter accounting for the underlying noncommutativity. In the present work, we will analyze the BPS self-dual solitons of the NC (relativistic) Chern-Simons-Higgs model in a field theoretic framework, following the second route. Our NC solitons have a smooth commutative limit such that for $\theta \rightarrow 0$ one recovers the commutative space-time solitons [2].

The method adapted here was exploited successfully by us in [13] (see also [16]) in the context of NC $CP(1)$ solitons [12]. The scheme utilizes the Seiberg-Witten map [6] to convert the NC action to an equivalent action in commutative space-time, comprised of ordinary field variables. The Seiberg-Witten map is crucial here since the theories in question are gauge theories in ordinary and

[1]This is probably dictated by the nature of involvement of the Higgs field in the soliton solution [15].

NC space-time. All the results derived here are valid to the first nontrivial order in θ. (This is mainly because there is some nonuniqueness in the Seiberg-Witten map in higher orders in θ.) Bogomolnyi analysis of the energy functional reveals the NC BPS solitons, which reduce smoothly in the $\theta \to 0$ to their commutative counterpart [2]. In order to check the validity of the expected equivalence between the two computational frameworks—operator and Moyal product—we can compare our result with that of Lozano *et. al.* in [11] since both have nonsingular commutative limits. However, it turns out that there is disagreement in the ''fine structure'' between the $O(\theta)$ result of Lozano *et. al.* [11] and our result. This underlines the importance of analyzing the same theory from different perspectives.

We study in detail various features of the NC solitons and come up with some surprising observations. Principal among them is the remarkable fact that, when the observables are suitably scaled, the $O(\theta)$ correction in the magnetic field (of the soliton) depends *only on θ and on no other parameters of the theory.* This indicates a sort of universality in the first order NC correction of the magnetic field, at least in these types of planar models. This is in contrast to the other characteristic features of the theory, such as the self-dual potential profile, the electric field or soliton energy, where, along with θ, another parameter τ (comprised solely of the commutative Chern-Simons-Higgs model parameters) plays an equally important role. Since, as such, τ is not restricted it is possible to have τ quite large so that the product $\theta\tau$ is not that small. On the other hand, as we will show, the freedom of choosing τ can be curbed somewhat via the requirement of the correct nature of the (modified Higgs) potential that can sustain soliton solutions.

Lastly we point out that compatibility between the BPS equation and the full variational equations of motion imposes restrictions on the form of the Higgs field and the parameters of the theory. This feature is present in the noncommutative extension only. We have also indicated circumstances when this restriction is not significant. This type of situation was previously reported in our analysis [13] of the NC $CP(1)$ soliton. This is probably connected to the proper definitions of the energy-momentum tensor (EMT) [17], from which the BPS equations originate. We have studied both the canonical and symmetric forms of the EMT and have shown that in the Hamiltonian framework, the correct space-time translation generators are reproduced from the canonical EMT. The expression for energy obtained from the symmetric EMT agrees with the canonical result modulo constraints. These subsidiary checks ensure that the EMT expressions are otherwise consistent. For this reason, the above-mentioned departure from the commutative (space-time) models, where solutions of the BPS equation automatically belong to the subset of solutions of the variational equation, is all the more unexpected and significant.

The paper is organized as follows: The NC $O(\theta)$ modified version of the Chern-Simons-Higgs model is introduced in Section II. The dynamical equations are derived in Section III. Section IV discusses the canonical EMT in a Hamiltonian framework. Section V deals with the constraint analysis and translation generators. Section VI comprises computation of the symmetric EMT. Section VII is devoted to the Bogomolnyi analysis and BPS equations. We have demonstrated the various NC effects pictorially in Section VIII. In Section IX we derive the condition on the Higgs field from the combination of the BPS and variational equations of motion. The paper ends with a conclusion in Section X.

II. NONCOMMUTATIVE CHERN-SIMONS-HIGGS MODEL

The space-time noncommutativity is governed by

$$[x^\rho, x^\sigma]_* = i\theta^{\rho\sigma}. \tag{2}$$

We restrict ourselves to only spatial noncommutativity ($\theta^{0i} = 0$) and the results are valid to the first nontrivial order in $\theta^{\mu\nu}$. The NC effect is encoded in the replacement of product of functions (in the action) by the associative $*$-product, given by the Moyal-Weyl formula,

$$p(x) * q(x) = pq + \frac{i}{2}\theta^{\rho\sigma}\partial_\rho p \partial_\sigma q + O(\theta^2). \tag{3}$$

The reason for invoking $\theta^{0i} = 0$ is that space-time noncommutativity can induce higher order time derivatives leading to a loss of causality in the field theory [18]. Also, even to $O(\theta)$, it can alter the symplectic structure in a significant way, that might result in a nonperturbative change in the dynamics, which we want to avoid.

In ordinary (commutative) space-time, the Chern-Simons-Higgs model[2] is described by the Lagrangian,

$$\mathcal{L} = \frac{\mu}{2}\epsilon^{\mu\nu\lambda}A_\mu\partial_\nu A_\lambda + \frac{1}{2}| D\phi|^2 - V(| \phi|^2) \tag{4}$$

where $D_\mu\phi = \partial_\phi - ieA_\mu\phi$. We will follow the procedure discussed in [3] where the form of $V(| \phi|^2)$ is kept arbitrary. It turns out that the BPS equations force it to be of a particular form. We will not repeat the derivation of the well-known soliton solutions of the commutative space-time [2,3] as they can be read off from our results by simply putting $\theta = 0$ (the limit being smooth). The NC counterpart of the above model is

[2]Our metric is $g^{\mu\nu} = diag(1, -1, -1)$.

$$\hat{\mathcal{L}} = \frac{\mu}{2}\epsilon^{\mu\nu\lambda}\Big(\hat{A}_\mu * \partial_\nu \hat{A}_\lambda + \frac{2}{3} i\hat{A}_\mu * \hat{A}_\nu * \hat{A}_\lambda\Big) + \frac{1}{2}(\hat{D}^\mu \hat{\phi})^* * \hat{D}_\mu \hat{\phi} - \hat{V}(|\phi|^2), \tag{5}$$

where $\hat{D}_\mu \hat{\phi} \equiv \partial_\mu \hat{\phi} - ie\hat{A}_\mu * \hat{\phi}$. Notice that the products in (4) are replaced by the $*$-product and the NC generalization of the Chern-Simons term is derived in [14]. $\hat{\psi}(x)$ is the NC counterpart of any generic field $\psi(x)$.

There are inequivalent ways of studying a generic NC field theory (or the model in (5) in particular) and the different approaches lead to distinct models. The conventional scheme [7–12] is the operator formalism where the fields are treated as operators. This is simply because they are functions of NC space-time arguments, the latter being elevated to operators in a Hilbert space since they obey NC commutation relations. The alternative scheme [13] is closer to conventional field theory (in commutative space-time) where the NC effects are taken into account in a perturbative way. Here explicit use of the Seiberg-Witten map [6] is made and generally only $O(\theta)$ effects are studied since the Seiberg-Witten map for higher orders in θ is not unique. However, since the Seiberg-Witten map is a *nonlinear* redefinition of field variables the resulting model can cease to be equivalent to the original one. Hence explicit forms of correlation functions can differ (due to the mapping) even though the effective actions at the level of partition function may remain unchanged. But as is common practice, we will refer to the Seiberg-Witten mapped model as the original one.

Exploiting the Seiberg-Witten map [6,19] to the first nontrivial order in θ,

$$\hat{\phi} = \phi - \frac{e}{2}\theta^{\alpha\beta}A_\alpha\partial_\beta\phi \qquad \hat{A}_\mu = A_\mu + \theta^{\sigma\rho}A_\rho\Big(\partial_\sigma A_\mu - \frac{1}{2}\partial_\mu A_\sigma\Big), \tag{6}$$

we recover the $O(\theta)$ corrected Lagrangian for the NC Chern-Simons-Higgs model,

$$\hat{\mathcal{L}} = \frac{\mu}{2}\epsilon^{\mu\nu\lambda}A_\mu\partial_\nu A_\lambda + \frac{1}{2}\Big[|D\phi|^2 + \frac{e}{2}\theta^{\alpha\beta} \times \Big\{F_{\alpha\mu}(D_\beta\phi^* D^\mu\phi + D^\mu\phi^* D_\beta\phi) - \frac{1}{2}F_{\alpha\beta}|D\phi|^2\Big\}\Big] - \hat{V}(|\phi|^2). \tag{7}$$

A simplified notation is used where $(D_\mu\phi)^* \equiv D_\mu\phi^* = (\partial_\mu + ieA_\mu)\phi^*$ and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the Abelian electromagnetic tensor.

We point out a feature of the particular NC extension (5) of the original model of (4). It should be remembered that the NC potential term $\hat{V}(|\phi|^2)$ in (7) is the NC extension of the (commutative) potential term in (4). Notice that so far it has been kept arbitrary, and its specific form will emerge from the Bogomolnyi analysis, just as in the original commutative case (see, for example, [3]). This is quite justified since we have only exploited the freedom of choosing the potential profile that can sustain soliton solutions in the NC case. However, no such freedom is available in the choice of the NC generalizations of the first two terms in (4) since one has to ensure $*$-gauge invariance in the NC counterpart. Thus, except for the potential term, the rest of the terms in (4) are generalized to NC space-time in the usual way and once again the form of $\hat{V}$ is kept arbitrary. As it turns out, the Bogomolnyi analysis will reveal that the NC extension of the potential is *not* obtainable from the original (commutative space-time) Higgs potential of [2] via the Seiberg-Witten map.

III. EQUATIONS OF MOTION

The Euler-Lagrange equations of motion for a generic field ψ_α,

$$\partial_\mu \frac{\delta\hat{\mathcal{L}}}{\delta(\partial_\mu\psi_\alpha)} - \frac{\delta\hat{\mathcal{L}}}{\delta\psi_\alpha} = 0, \tag{8}$$

yields the following dynamical equations for the model in question (7),

$$\frac{\mu}{2}\epsilon^{\sigma\alpha\beta}F_{\alpha\beta} + \frac{ie}{2}(\phi^* D^\sigma\phi - \phi D^\sigma\phi^*) = \frac{e}{4}\partial_\rho[\theta^{\rho\beta}(D_\beta\phi^* D^\sigma\phi + D^\sigma\phi^* D_\beta\phi) - \theta^{\sigma\beta}(D_\beta\phi^* D^\rho\phi + D^\rho\phi^* D_\beta\phi) - \theta^{\rho\sigma}|D\phi|^2] - \frac{ie^2}{4}\Big[\theta^{\alpha\sigma}F_{\alpha\mu}(\phi^* D^\mu\phi - \phi D^\mu\phi^*) + \theta^{\alpha\beta}F^\sigma_\alpha \times (\phi^* D_\beta\phi - \phi D_\beta\phi^*) - \frac{1}{2}\theta^{\alpha\beta}F_{\alpha\beta}(\phi^* D^\sigma\phi - \phi D^\sigma\phi^*)\Big], \tag{9}$$

$$\frac{1}{2}D_\mu D^\mu\phi + \frac{\delta\hat{V}}{\delta\phi^*} = -\frac{e}{4}\partial_\rho[\theta^{\alpha\rho}F_{\alpha\mu}D^\mu\phi + \theta^{\alpha\beta}F^\rho_\alpha D_\beta\phi] + i\frac{e^2}{4}\theta^{\alpha\beta}\Big[F_{\alpha\mu}(A_\beta D^\mu\phi + A^\mu D_\beta\phi) - \frac{1}{2}F_{\alpha\beta}A_\mu D^\mu\phi\Big]. \tag{10}$$

We now restrict ourselves to only spatial noncommutativity, i.e., $\theta_{0i} = 0$ and define $\theta_{ij} \equiv \epsilon_{ij}\theta$, $F_{ij} \equiv \epsilon_{ij}F$. This leads us to the following (manifestly) noncovariant equations corresponding to (9),

$$\mu F = -\frac{ie}{2}(\phi^* D_0\phi - \phi D_0\phi^*)\Big(1 - \frac{e\theta F}{2}\Big) + \frac{e\theta}{4}\epsilon_{ij}\{\partial_i(D_j\phi^* D_0\phi + D_j\phi D_0\phi^*) - ieF_{i0}(\phi^* D_j\phi - \phi D_j\phi^*)\} \quad (11)$$

$$\mu\epsilon_{ij}F_{j0} = \frac{ie}{2}(\phi^* D_i\phi - \phi D_i\phi^*)\Big(1 + \frac{e\theta F}{2}\Big) + \frac{e\theta}{4}\epsilon_{ij}\{-\partial_0(D_j\phi^* D_0\phi + D_j\phi D_0\phi^*) + \partial_j(D_0\phi^* D_0\phi) + ieF_{j0}(\phi^* D_0\phi - \phi D_0\phi^*)\}. \quad (12)$$

We will often use the above equations in the form,

$$\mu F \approx -\frac{ie}{2}(\phi^* D_0\phi - \phi D_0\phi^*) + O(\theta), \quad (13)$$

$$\mu\epsilon_{ij}F_{j0} \approx \frac{ie}{2}(\phi^* D_i\phi - \phi D_i\phi^*) + O(\theta). \quad (14)$$

IV. HAMILTONIAN ANALYSIS AND CANONICAL ENERGY-MOMENTUM TENSOR

Let us now introduce the Hamiltonian formulation of the problem at hand. Our aim is to obtain the space-time and gauge symmetry generators and subsequently study the (space-time and gauge) transformation properties of the fields. A similar kind of analysis has been done for the $CP(1)$ model in [16]. This requires the construction of the canonical EMT,

$$T_c^{\rho\nu} \equiv \frac{\delta\hat{\mathcal{L}}}{\delta(\partial_\rho A_\sigma)}\partial^\nu A_\sigma + \frac{\delta\hat{\mathcal{L}}}{\delta(\partial_\rho\phi)}\partial^\nu\phi + \frac{\delta\hat{\mathcal{L}}}{\delta(\partial_\rho\phi^*)}\partial^\nu\phi^*. \quad (15)$$

In the present case, we get the canonical EMT

$$T_c^{\rho\nu} = \Big[\frac{\mu}{2}\epsilon^{\mu\rho\sigma}A_\mu + \frac{e}{4}\{\theta^{\rho\beta}(D_\beta\phi^* D^\sigma\phi + D^\sigma\phi^* D_\beta\phi) - \theta^{\sigma\beta}(D_\beta\phi^* D^\rho\phi + D^\rho\phi^* D_\beta\phi) - \theta^{\rho\sigma}|D\phi|^2\}\Big]\partial^\nu A_\sigma + \xi^{\rho\nu} + (\xi^{\rho\nu})^* - g^{\rho\nu}\hat{\mathcal{L}}, \quad (16)$$

where

$$\xi^{\rho\nu} = \Big[\frac{1}{2}\Big(1 - \frac{e\theta F}{2}\Big)D^\rho\phi^* + \frac{e}{4}(\theta^{\alpha\rho}F^\beta_\alpha D_\beta\phi^* + \theta^{\alpha\beta}F^\rho_\alpha D_\beta\phi^*)\Big]\partial^\nu\phi.$$

The energy and momentum densities follow immediately:

$$T^c_{00} = \hat{V} - \mu A_0 F + \frac{1}{2}\Big(1 - \frac{e\theta F}{2}\Big)|D_0\phi|^2 + \frac{1}{2}\Big(1 + \frac{e\theta F}{2}\Big)|D_k\phi|^2 + \frac{ie}{2}\Big(1 - \frac{e\theta F}{2}\Big)A_0(\phi D_0\phi^* - \phi^* D_0\phi) + \frac{e\theta}{4}\epsilon_{kj}[F_{k0}\{D_j\phi^* D_0\phi + D_j\phi D_0\phi^* + ieA_0(\phi D_j\phi^* - \phi^* D_j\phi)\} - \partial_k A_0(D_j\phi^* D_0\phi + D_j\phi D_0\phi^*)], \quad (17)$$

$$T_c^{0i} = \frac{\mu}{2}\epsilon_{jk}A_j\partial_i A_k - \frac{1}{2}\Big(1 - \frac{e\theta F}{2}\Big)(D_0\phi^*\partial_i\phi + D_0\phi\partial_i\phi^*) - \frac{e\theta}{4}\epsilon_{jk}(D_j\phi^* D_0\phi + D_j\phi D_0\phi^*)\partial_i A_k - \frac{e\theta}{4}\epsilon_{jk}F_{k0}(D_k\phi^*\partial_i\phi + D_k\phi\partial_i\phi^*). \quad (18)$$

The next task is to introduce the canonical momenta which will indicate that the theory has constraints and hence a Hamiltonian constraint analysis becomes imperative. Defining the momenta as

$$\pi^* \equiv \frac{\delta\mathcal{L}}{\delta\dot{\phi}^*}, \qquad \pi \equiv \frac{\delta\mathcal{L}}{\delta\dot{\phi}}, \qquad \pi^\mu \equiv \frac{\delta\mathcal{L}}{\delta\dot{A}_\mu}$$

we find,

$$\begin{aligned} \pi^* &= \frac{1}{2}\Big(1 - \frac{e\theta}{2}\Big)D_0\phi + \frac{e\theta}{4}\epsilon_{ij}F_{i0}D_j\phi \approx \frac{1}{2}D_0\phi + O(\theta), \\ \pi &= \frac{1}{2}\Big(1 - \frac{e\theta}{2}\Big)D_0\phi^* + \frac{e\theta}{4}\epsilon_{ij}F_{i0}D_j\phi^* \approx \frac{1}{2}D_0\phi^* + O(\theta), \qquad \pi_0 = 0; \\ \pi_k &= \frac{\mu}{2}\epsilon_{kj}A_j - \frac{e\theta}{4}\epsilon_{kj}(D_j\phi^* D_0\phi + D_j\phi D_0\phi^*) \approx \frac{\mu}{2}\epsilon_{kj}A_j + O(\theta). \end{aligned} \quad (19)$$

This allows us to rewrite the above defining relations in (19) to $O(\theta)$ in the following way:

$$\pi^* = \frac{1}{2}D_0\phi - \frac{e\theta}{2}F\pi^* + \frac{e\theta}{4}\epsilon_{ij}F_{i0}D_j\phi + O(\theta^2), \qquad \pi = \frac{1}{2}D_0\phi^* - \frac{e\theta}{2}F\pi + \frac{e\theta}{4}\epsilon_{ij}F_{i0}D_j\phi^* + O(\theta^2), \quad (20)$$

$$\pi_k = \frac{\mu}{2}\epsilon_{kj}A_j - \frac{e\theta}{2}\epsilon_{kj}(\pi^* D_j\phi^* + \pi D_j\phi) + O(\theta^2). \tag{21}$$

We also invert the above relations to $O(\theta)$ and get

$$\begin{aligned} D_0\phi &= 2\pi^* + e\theta\left(F\pi^* - \frac{1}{2}\epsilon_{ij}F_{i0}D_j\phi\right) + O(\theta^2),\\ D_0\phi^* &= 2\pi + e\theta\left(F\pi - \frac{1}{2}\epsilon_{ij}F_{i0}D_j\phi^*\right) + O(\theta^2). \end{aligned} \tag{22}$$

Reexpressed in terms of the phase space variable the total energy and momenta look like,

$$\begin{aligned} H_c &= \int d^2x T_c^{00} \\ &= \int d^2x\left[2\left(1 + \frac{e\theta F}{2}\right)\left(\pi^*\pi + \frac{1}{4}D_i\phi^* D_i\phi\right) \right.\\ &\quad \left. + \hat{V} + A_0\mathcal{G}\right], \end{aligned} \tag{23}$$

$$P_i^c = \int d^2x T_{0i}^c = \int d^2x[\pi\partial_i\phi + \pi^*\partial_i\phi^* + \pi_j\partial_i A_j], \tag{24}$$

where

$$\mathcal{G} \equiv -\partial_i\pi_i - \frac{\mu}{2}F + ie(\phi\pi - \phi^*\pi^*) \approx 0 \tag{25}$$

is the Gauss law constraint besides the trivial one, $\pi_0 \approx 0$.

V. CONSTRAINTS, DIRAC BRACKETS, AND TRANSLATION GENERATORS

The two constraints, $\mathcal{G} \approx 0$ and $\pi_0 \approx 0$, constitute the First Class Constraints (FCCs) of the present theory in the terminology of Dirac [20]. The FCCs commute (in the sense of Poisson brackets) and signify local gauge invariance (which is U(1) in the present case).

Besides the above FCCs, following from the relation (21) there are furthermore two Second Class Constraints (SCC) [20],

$$\chi_i \equiv \pi_i - \frac{\mu}{2}\epsilon_{ij}A_j(1-P) + \frac{e\theta}{2}\epsilon_{ij}(\pi\partial_j\phi + \pi^*\partial_j\phi^*), \tag{26}$$

where

$$P \equiv -\frac{i\theta e^2}{\mu}(\phi\pi - \phi^*\pi^*).$$

The SCCs are noncommuting (in the sense of Poisson brackets) and they induce a change in the symplectic structure, whereby a generic Poisson bracket ($\{A, B\}$) is replaced by a Dirac bracket ($\{A, B\}_{DB}$), defined in the following way,

$$\{A, B\}_{DB} = \{A, B\} - \{A, \chi_i\}\chi_{ij}^{-1}\{\chi_j, B\}. \tag{27}$$

In the above definition (27), χ_{ij}^{-1} denotes the inverse of the Poisson bracket matrix,

$$\chi_{ij}(x, Y) \equiv \{\chi_i(x), \chi_j(y)\} = -\mu\epsilon_{ij}(1-P)\delta(x-y), \tag{28}$$

where $P = (i\theta e^2/\mu)(\pi^*\phi^* - \pi\phi)$. The inverse is computed to be

$$\chi_{jk}(x, y)^{-1} = \frac{1}{\mu}\epsilon_{jk}(1+P)\delta(x-y) + O(\theta^2). \tag{29}$$

Utilizing the defining Eq. (27), it is now straightforward to obtain the full set of Dirac brackets which are given below:

$$\begin{aligned} &\{A_i(x), A_j(y)\} = \frac{1}{\mu}\epsilon_{ij}(1+P)\delta(x-y); \quad \{A_i(x), \pi_j(y)\} = \frac{1}{2}\delta_{ij}\delta(x-y); \quad \{A_i(x), \phi(y)\} = -\frac{e\theta}{2\mu}D_i\phi\delta(x-y);\\ &\{A_i(x), \pi(y)\} = \frac{e\theta}{2\mu}\pi(x)D_i^{(x)}\delta(x-y); \quad \{\pi_i(x), \pi_j(y)\} = \frac{\mu}{4}(1-P)\epsilon_{ij}\delta(x-y);\\ &\{\pi_i(x), \phi(y)\} = \frac{e\theta}{4}\epsilon_{ij}D_j\phi\delta(x-y); \quad \{\pi_i(x), \pi(y)\} = -\frac{e\theta}{4}\epsilon_{ij}\pi(x)D_j^{(x)}\delta(x-y), \end{aligned} \tag{30}$$

$$\begin{aligned} &\{\phi(x), \pi(y)\} = \delta(x-y) + O(\theta^2);\\ &\{\phi(x), \phi(y)\} = \{\pi(x), \pi(y)\} = O(\theta^2). \end{aligned} \tag{31}$$

It should be remembered that *all* the above relations are valid up to $O(\theta)$. Notice that in this approximation, there is no modification in the $\phi - \pi$ sector. Also note that, starting from the set of relations (30) above, and in the subsequent discussion, unless otherwise stated, *all* the brackets are Dirac brackets and so we have dropped the subscript $\{,\}_{DB}$ from now on.

Our first objective is to apply the Dirac brackets to ensure that the fields are transforming in the proper way under the symmetry transformations. We start with gauge transformation, the infinitesimal transformation generator of which is given by

$$G \equiv \int d^2x\lambda(x)\mathcal{G}(x), \tag{32}$$

$\lambda(x)$ being the infinitesimal parameter. Using the Dirac brackets (30) and (31), we find

$$\{\phi(x), \mathcal{G}(y)\} = ie\phi(x)\delta(x-y) \rightarrow \{\phi(x), G\} = ie\lambda(x)\phi(x), \quad (33)$$

$$\{A_i(x), \mathcal{G}(y)\} = \partial_i^{(x)}\delta(x-y) \rightarrow \{A_i(x), G\} = \partial_i\lambda(x). \quad (34)$$

Thus the gauge properties of the charged scalar field ϕ and the U(1) gauge field A_i are correctly recovered.

Next we study the space-time transformation properties of the fields. It is now a simple task to establish the following relation,

$$\{\psi(x), P_i^c\} = \partial_i\psi(x), \quad (35)$$

where $\psi \equiv \{\phi, \pi, A_i, \pi_i\}$. The expression for P_i^c is given in (24) and the new symplectic structure (30) and (31), is used. This indicates that the momentum operator P_i^c correctly plays the role of the generator of spatial translation. From the explicit form of P_i^c given in (24) it is evident that the translation generator is essentially canonical and that the noncommutativity has not generated any extra contribution. This feature obviously reflects the translation invariance of the starting model (7). An identical situation prevailed in [16].

However, recovering the Hamiltonian form of the equations of motion, which is equivalent to obtaining the time derivatives correctly, turns out to be somewhat nontrivial. The following bracket,

$$\{\phi(x), H^c\} = D_0\phi(x), \quad (36)$$

indicates that the proper definition of time derivative for ϕ, i.e.,

$$\{\phi(x), H^c\} = \partial_0\phi(x), \quad (37)$$

requires a gauge fixing $A_0 \approx 0$. Indeed, this gauge choice is harmless as far as the Dirac brackets are concerned since it simply removes the SCCs π_0 and A_0 (that constitutes a canonically conjugate pair) from further considerations without affecting the rest of the brackets. Considering A_i we find

$$\{A_i(x), H^c\} = -F_{i0}\left(1 + \frac{e\theta}{\mu}\mathcal{G}\right) - \frac{e\theta}{4\mu}[D_i\phi^* D_\mu D^\mu\phi + D_i\phi D_\mu^*(D^\mu\phi)^* - \partial_i(\partial_0\phi^*\partial_0\phi)] - \frac{e\theta}{2\mu}\partial_i\hat{V}. \quad (38)$$

In deriving the above relation, we have used $A_0 = 0$ gauge. Exploiting the dynamical equation for ϕ from (10),

$$D_\mu D^\mu\phi = -2\frac{\delta\hat{V}}{\delta\phi^*} + O(\theta);$$
$$D_\mu^*(D^\mu\phi)^* = -2\frac{\delta\hat{V}}{\delta\phi} + O(\theta),$$

in the above equation, we find a simplified relation,

$$\{A_i(x), H^c\} = \partial_0 A_i + \frac{e\theta}{2\mu}\partial_i(\partial_0\phi^*\partial_0\phi) + O(\theta^2). \quad (39)$$

Notice that there still remains an $O(\theta)$ extra piece. However, it has the structure of a U(1) gauge transformation. Since the Gauss law FCC is still intact, we are allowed to make a further gauge transformation thus maintaining the proper definition of a time derivative. This feature is reminiscent of gauge theories in commutative space-time, where the gauge field A_i behaves properly under Lorentz boosts modulo a gauge transformation. It should be remembered that in our study [16] of the $CP(1)$ model, also, deriving the Hamiltonian equations of motion turned out to be more involved.

VI. SYMMETRIC ENERGY-MOMENTUM TENSOR

Let us now construct the symmetric form of the EMT that is to be utilized in obtaining the BPS soliton solutions of the NC Chern-Simons-Higgs model. This has been the common practice in existing literature for commutative [2,3] and noncommutative [11] Chern-Simons-Higgs solitons. The symmetric form of the EMT is conventionally defined as [17]

$$T_{\mu\nu}^s \equiv \frac{2}{\sqrt{-g}}\frac{\delta\mathcal{L}}{\delta g^{\mu\nu}}. \quad (40)$$

In the present case this is obtained by coupling the model with the metric field $g_{\mu\nu}$ (with $g_{\mu\nu}$ being a background field) and finally reducing it to a flat Minkowski background. This generates the symmetric EMT,

$$T_{\mu\nu}^s = -g_{\mu\nu}\bar{\mathcal{L}} + \frac{1}{2}(D_\mu\phi^* D_\nu\phi + D_\nu\phi^* D_\mu\phi)\left(1 - \frac{e\theta F}{2}\right) - \frac{e}{4}D_\beta\phi^* D^\beta\phi(\theta_{\mu\alpha}F_\nu^\alpha + \theta_{\nu\alpha}F_\mu^\alpha) + \frac{e}{4}[(\theta_{\mu\alpha}F_{\nu\beta} + \theta_{\nu\alpha}F_{\mu\beta}) \times (D^\alpha\phi^* D^\beta\phi + D^\beta\phi^* D^\alpha\phi) + F^{\alpha\beta}\{\theta_{\alpha\mu}(D_\nu\phi^* D_\beta\phi + D_\beta\phi^* D_\nu\phi) + \theta_{\alpha\nu}(D_\mu\phi^* D_\beta\phi + D_\beta\phi^* D_\mu\phi)\} + \theta^{\alpha\beta}\{F_{\alpha\mu}(D_\beta\phi^* D_\nu\phi + D_\nu\phi^* D_\beta\phi) + F_{\alpha\nu}(D_\beta\phi^* D_\mu\phi + D_\mu\phi^* D_\beta\phi)\}]. \quad (41)$$

In the above expression, $\bar{\mathcal{L}}$ represents the Lagrangian (7) without the Chern-Simons term since the topological term is metric independent and does not contribute in the variation of the metric tensor.

It is straightforward to check that for $\theta = 0$, the symmetric EMT is conserved,

$$\partial^\mu T^s_{\mu\nu} \mid_{\theta=0} = 0.$$

Although the same is not true for the $O(\theta)$ corrected $T^s_{\mu\nu}$, we have explicitly checked that a modified EMT can be defined in the present case which is symmetric and conserved (see Das and Frenkel in [17]).

Now (41) leads to the following expression for the energy density,

$$\begin{aligned} T^s_{00} &= \frac{1}{2}(1-\alpha)D_0\phi^* D_0\phi + \frac{1}{2}(1+\alpha)D_i\phi^* D_i\phi \\ &\quad + \frac{e\theta}{4}\epsilon_{ij}F_{i0}(D_j\phi^* D_0\phi + D_j\phi D_0\phi^*) + \hat{V} \\ &= \frac{1}{2}(1-\alpha)D_0\phi^* D_0\phi + \frac{1}{2}(1+\alpha)D_i\phi^* D_i\phi \\ &\quad - \frac{i\theta e^2}{8\mu}(\phi^* D_j\phi - \phi D_j\phi^*) \\ &\quad \times (D_j\phi^* D_0\phi + D_j\phi D_0\phi^*) + \hat{V}, \end{aligned} \tag{42}$$

where $\alpha = \frac{e\theta F}{2}$ and the equations of motion (12) have been used.

As a curiosity, let us express the Hamiltonian and momenta obtained from the symmetric EMT in terms of the phase space variables defined before in (19). We find

$$\int d^2x T^{00}_s = \int d^2x\left[2\left(1+\frac{e\theta F}{2}\right) \times\left(\pi^*\pi + \frac{1}{4}D_i\phi^* D_i\phi\right) + \hat{V}\right], \tag{43}$$

which agrees with the canonical form H_c given in (23) on the constraint surface. The expression for the (symmetric) momentum density as a function of phase space degrees of freedom is

$$\begin{aligned} T^s_{0i} = T^s_{i0} &= (\pi D_i\phi + \pi^* D_i\phi^*)(1 + e\theta F) \\ &\quad + \frac{i\theta e^2}{8\mu}\Big[4(\phi^* D_i\phi - \phi D_i\phi^*) \\ &\quad \times\left(\mid\pi\mid^2 + \frac{1}{4}\mid D_i\phi\mid^2\right) + (\phi^* D_j\phi - \phi D_j\phi^*) \\ &\quad \times (D_j\phi^* D_i\phi + D_j\phi D_i\phi^*)\Big]. \end{aligned} \tag{44}$$

This is quite distinct from the canonical expression of momentum obtained in (24).

In this context let us note a puzzling feature regarding the conservation of the energy. We have shown in (43) that in terms of phase space variables, the energy obtained from the symmetric EMT agrees with the energy obtained from the canonical EMT. In the previous section, we have also shown that this Hamiltonian correctly generates time translations and hence it should be conserved as well. This does not appear to be valid in the Lagrangian picture.

VII. BOGOMOLNYI ANALYSIS AND BPS EQUATIONS FOR SOLITON

Our approach is the same as that of the commutative space-time Bogomolnyi analysis [2,3]. Similar analysis for the NC $CP(1)$ model was performed in [13]. We concentrate on the energy expression provided in (42) and write it in the form,

$$\begin{aligned} H^s &= \int d^2x T^s_{00}(x) = \int \frac{1}{2}(1+\alpha)\mid D_\pm\phi\mid^2 + \frac{1}{2}\left|\sqrt{1-\alpha}D_0\phi \pm \frac{ie^2}{2\mu}\sqrt{1+\alpha}\right. \\ &\quad \left.\times(\mid\phi\mid^2 - v^2)\phi - \frac{\theta e^2}{4\mu}\gamma_j D_j\phi\right|^2 \mp \frac{ev^2}{2\mu}J_0 \pm \frac{e\alpha}{2\mu}(\mid\phi\mid^2 - v^2)J_0 + \hat{V} - \frac{e^4}{8\mu^2}(1+\alpha)(\mid\phi\mid^2 - v^2)^2 \\ &\quad \times\mid\phi\mid^2 \mp \frac{1}{2}(e\alpha\epsilon_{ij}A_j\partial_i\mid\phi\mid^2 + i\alpha\epsilon_{ij}\partial_i\phi^*\partial_j\phi) \mp \frac{\theta e^4}{16\mu^2}(\mid\phi\mid^2 - v^2)(\phi^* D_j\phi - \phi D_j\phi^*)^2 \mp \frac{i\theta e^4}{16\mu^2} \\ &\quad \times(\mid\phi\mid^2 - v^2)(\phi^* D_j\phi - \phi D_j\phi^*)\gamma_j, \end{aligned} \tag{45}$$

where

$$D_\pm \equiv D_1 \pm iD_2; \qquad \gamma_j \equiv \pm\epsilon_{ij}\partial_i\mid\phi\mid^2 + i(\phi^* D_j\phi - \phi D_j\phi^*).$$

The following identities have been exploited:

$$\begin{aligned} \int d^2x\frac{1}{2}D_i\phi^* D_i\phi &= \int d^2x\left[\frac{1}{2}\mid D_\pm\phi\mid^2 \pm \frac{e}{2}F\mid\phi\mid^2\right]\int d^2x\frac{1}{2}\alpha D_i\phi^* D_i\phi \\ &= \int d^2x\left[\alpha\left(\frac{1}{2}\mid D_\pm\phi\mid^2 \pm \frac{e}{2}F\mid\phi\mid^2\right) \mp \frac{1}{2}(e\alpha F\mid\phi\mid^2 + e\alpha\epsilon_{ij}A_j\partial_i\mid\phi\mid^2 + i\alpha\epsilon_{ij}\partial_i\phi^*\partial_j\phi)\right]. \end{aligned} \tag{46}$$

Also note that similar to the commutative case [3] we have defined the *conserved* U(1) current from (9) in the form,

$$\epsilon^{\mu\nu\lambda}F_{\nu\lambda} \equiv -\frac{2}{\mu}J^{\mu} \Rightarrow \partial_{\mu}J^{\mu} = 0. \quad (47)$$

This means that our expression for the conserved current contains $O(\theta)$ terms and J_0 is the conserved charge density. However, defining the current in this way ensures that the charge-flux equality will still remain intact. In particular, in arriving at (45), we have used the relation

$$\frac{ie}{2}(\phi^* D_0\phi - \phi D_0\phi^*) = (1+\alpha)J_0 - \frac{\theta e^3}{8\mu} \times (\phi^* D_j\phi - \phi D_j\phi^*)^2 + \frac{\theta e}{4}\epsilon_{ij}\partial_i(D_j\phi^* D_0\phi + D_j\phi D_0\phi^*). \quad (48)$$

It can be checked that upon simplification, all but one line of (45) vanishes identically. Let us now make the ansatz that

$$D_{\pm}\phi = O(\theta) \Rightarrow D_i\phi = \pm i\epsilon_{ij}D_j\phi + O(\theta). \quad (49)$$

This is justified since we expect $O(\theta)$ modifications in the results pertaining to commutative space-time ($\theta = 0$). This immediately leads to

$$\gamma_j = O(\theta). \quad (50)$$

Hence one can ignore the terms containing γ_j since they are always multiplied by θ. This simplifies the situation considerably and we are led to the cherished BPS equations for the solitons of the NC Chern-Simons-Higgs theory, in the lowest nontrivial order in θ:

$$D_{\pm}\phi = 0 \Rightarrow D_i\phi = \pm i\epsilon_{ij}D_j\phi, \quad (51)$$

$$D_0\phi \pm \frac{ie^2}{2\mu}(1+\alpha)(|\phi|^2 - v^2)\phi = 0, \quad (52)$$

with γ_j vanishing on the BPS shell. This constitutes our main result. The analysis determines the potential profile to be

$$\hat{V} = \frac{e^4}{8\mu^2}(1+\alpha)(|\phi|^2 - v^2)^2 |\phi|^2. \quad (53)$$

Let us choose the *lower* sign in the BPS Eqs. (51) and (52), which we refer to as the self-dual solution. The upper sign will correspond to the anti-self-dual solution.

We want to express the gauge-field quantities in terms of the ϕ fields by exploiting the BPS equations. This is convenient since the BPS equations are first order in derivatives.

First of all, from the equation of motion (11) we obtain F as

$$F \equiv -\frac{J_0}{\mu} = -\frac{ie}{2\mu}(\phi^* D_0\phi - \phi D_0\phi^*) + \frac{\theta e^3}{8\mu^2}[(\phi^* D_0\phi - \phi D_0\phi^*)^2 - (\phi^* D_i\phi - \phi D_i\phi^*)^2] + \frac{\theta e}{4\mu}\epsilon_{ij}\partial_i(D_j\phi^* D_0\phi + D_j\phi D_0\phi^*). \quad (54)$$

Substituting the BPS equations in the above relation we get, for the self-dual case,

$$F_{sd} = \frac{e^3}{2\mu^2}(|\phi|^2 - v^2)|\phi|^2 - \frac{\theta e^3}{8\mu^2}(|\phi|^2 - v^2)\nabla|\phi|^2, \quad (55)$$

where $\nabla \equiv \partial_i\partial_i$. Putting this back in the expression for the energy (45) we get the BPS saturated energy lower bound (or, equivalently, the soliton mass) as

$$W_{sd} = \int\left[\frac{ev^2}{2\mu}J_0 - \frac{\theta e^2}{4\mu^2}(v^2 - \phi^2)J_0^2\right] = \int\left[-\frac{ev^2}{2}F - \frac{\theta e^2}{4}(v^2 - \phi^2)F^2\right]. \quad (56)$$

As a final step, we can express the potential and energy completely in terms of ϕ:

$$\hat{V}_{sd} = \frac{e^4}{8\mu^2}\left[(|\phi|^2 - v^2)^2|\phi|^2 + \theta\frac{e^4}{4\mu^2}(|\phi|^2 - v^2)^3(|\phi|^2)^2\right], \quad (57)$$

$$W_{sd} = \int\frac{e^4 v^2}{4\mu^2}\left[-(|\phi|^2 - v^2)|\phi|^2 + \theta\left\{\frac{e^4}{4\mu^2 v^2}(|\phi|^2 - v^2)^3(|\phi|^2)^2 + \frac{1}{4}(|\phi|^2 - v^2)\nabla|\phi|^2\right\}\right]. \quad (58)$$

The electric field is obtained in the form

$$(E_k)_{sd} \equiv (F_{k0})_{sd} = -\frac{e}{2\mu}\partial_k|\phi|^2 + \frac{\theta e^5}{16\mu^3}(|\phi|^2 - v^2) \times (3|\phi|^2 - v^2)\partial_k|\phi|^2, \quad (59)$$

from which we get the magnitude of the electric field,

$$(E_k E_k)_{sd} = \frac{e^2}{4\mu^2}\partial_k \mid \phi|^2 \partial_k \mid \phi|^2 \times -\frac{\theta e^6}{16\mu^4}\partial_k \mid \phi|^2 \partial_k \mid \phi|^2(\mid \phi|^2 - v^2) \times (3 \mid \phi|^2 - v^2). \quad (60)$$

By choosing a simple time dependence of the ϕ field [3], such as

$$\phi(x, t) = e^{-i(e^2 v^2/2\mu)t}\varphi(x), \quad (61)$$

$(A_0)_{sd}$ is identified as

$$(A_0)_{sd} = -\frac{e}{2\mu} \mid \phi|^2\left[1 + \frac{\theta e^4}{4\mu^2}(\mid \phi|^2 - v^2)^2\right]. \quad (62)$$

Analogous expressions for the anti-self-dual soliton solutions are

$$\hat{V}_{asd} = \frac{e^4}{8\mu^2}\left[(\mid \phi|^2 - v^2)^2 \mid \phi|^2 - \theta\frac{e^4}{4\mu^2} \times (\mid \phi|^2 - v^2)^3(\mid \phi|^2)^2\right], \quad (63)$$

$$F_{asd} = -\frac{e^3}{2\mu^2}(\mid \phi|^2 - v^2) \mid \phi|^2 - \frac{\theta e^3}{8\mu^2} \times (\mid \phi|^2 - v^2)\nabla \mid \phi|^2, \quad (64)$$

$$(E_k E_k)_{asd} = \frac{e^2}{4\mu^2}\partial_k \mid \phi|^2 \partial_k \mid \phi|^2 - \frac{\theta e^6}{16\mu^4}\partial_k \mid \phi|^2 \partial_k \times \mid \phi|^2[(\mid \phi|^2)^2 - v^4], \quad (65)$$

$$W_{asd} = \int \frac{e^4 v^2}{4\mu^2}\left[-(\mid \phi|^2 - v^2) \mid \phi|^2 - \theta\left\{\frac{e^4}{4\mu^2 v^2}(\mid \phi|^2 - v^2)^3(\mid \phi|^2)^2 + \frac{1}{4}(\mid \phi|^2 - v^2)\nabla \mid \phi|^2\right\}\right]. \quad (66)$$

As an aside, it is interesting to observe that, using the Seiberg-Witten map for the ϕ variables,

$$\hat{\phi}^*\hat{\phi} = \phi^*\phi + \frac{1}{2}\theta^{\alpha\beta}[i\partial_\alpha\phi^*\partial_\beta\phi + eA_\beta\partial_\alpha \mid \phi|^2] + O(\theta^2), \quad (67)$$

it is not possible to correctly generate the $O(\theta)$ term in the potential in (57) from the θ-independent potential term. The latter, being the Higgs potential for commutative space-time, can be obtained from the NC potential by putting $\theta = 0$. This phenomenon indicates that the NC extension of the Chern-Simons-Higgs soliton is not derivable by just applying the Seiberg-Witten map on the commutative model. The potential has to be tuned properly.

VIII. EFFECTS OF NONCOMMUTATIVITY ON THE SOLITON

We now demonstrate the effects of noncommutativity on the soliton. It is quite remarkable that the electric and magnetic fields of the soliton are affected in very *different* ways. In fact, the parameters of the Chern-Simons-Higgs model (e, μ, and v) enter in the fray in a nontrivial way. This can be noticed after appropriate scaling of the observables as we now illustrate. There appears to be a universal (i.e., parameter independent) nature in the noncommutative effects in the magnetic field as it depends only on θ. However, this property is not shared by the electric field or the potential, where θ as well as the combination $\tau \equiv e^4 v^4/8\mu^2$ play equally important roles. Note that τ consists entirely of parameters of the (commutative) Chern-Simons-Higgs model and as such is not restricted by any bounds. However, in the presence of noncommutativity, formation of the potential well demands certain bounds on the value of τ. These features were not reported in the earlier literature [11].

We now move on to the axially symmetric solutions where n elementary vortices are superimposed at the origin. As has been discussed before [13,16], the spatial noncommutativity in 2 + 1-dimensions does not destroy the rotational symmetry. (This is also evident from the canonical nature of the momentum operator discussed here.) Notice that in (51) and (52) the noncommutativity has modified only part of the full set of BPS equations. We try solutions of the form

$$\phi = vg(r)e^{in\phi}; \qquad eA_i = \epsilon_{ij}\frac{\hat{r}_j}{r}[a(r) - n], \quad (68)$$

with $F = -\frac{a'}{er}$ and (r, ϕ) denoting the plane polar coordinates. This brings us to the set of consistency conditions,

$$-\frac{a'}{er} = \frac{e^3 v^4}{2\mu^2}\left[g^2(g^2 - 1) - \frac{\theta}{4}(g^2 - 1)\nabla g^2\right]. \quad (69)$$

This shows that to linear order in θ we can use the same expression for $g(r)$ as given in [2] but $a(r)$ requires a θ-correction term, which can also be expressed in terms of $g(r)$. The scenario can be compared with the commutative case [2]. From now on we will exploit only the form of $g(r)$ with the same boundary conditions as given in [2] and consider the single soliton case, i.e., $n = 1$.

In Figs. 1(a), 1(b), and 2, we have shown the noncommutativity effect on the potential. We have plotted

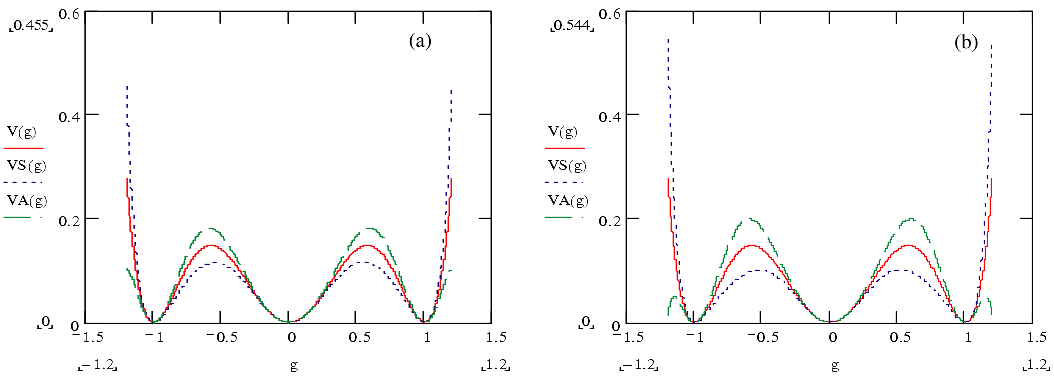

FIG. 1 (color online). We plot the potentials VS (for self-dual) and VA (for anti-self-dual) against g for $\theta = 0.5$ and $\tau = 0.5$ and $\tau = 1.5$ in (a) and (b), respectively. This shows that for the anti-self-dual case, there is a critical value of τ above which the double well will not be formed. For the self-dual case, the potential flattens out for large τ as shown in Fig. 2.

$$V_{sd}(g)/\left(\frac{e^4 v^6}{8\mu^2}\right) \equiv VS(g) = (g^2-1)g^2 + 2\theta\tau(g^2-1)^3 g^4,$$

$$V_{asd}(g)/\left(\frac{e^4 v^6}{8\mu^2}\right) \equiv VA(g) = (g^2-1)g^2 - 2\theta\tau(g^2-1)^3 g^4, \quad (70)$$

with $V(g)$ representing the $\theta = 0$ case. The points to notice are: (i) The Chern-Simons-Higgs parameter τ appears in the expressions. (ii) The correction term tends to flatten the potential humps in the self-dual case (see Fig. 2). (iii) For the anti-self-dual case, the well structure can disappear when some critical value of the $\theta\tau$ combination is reached [see Fig. 1(b)]. Since θ is assumed to be small, bounds can be put on the value on the scale of τ.

In Fig. 3(a)–3(c), we have plotted the magnetic field F as a function of r given by

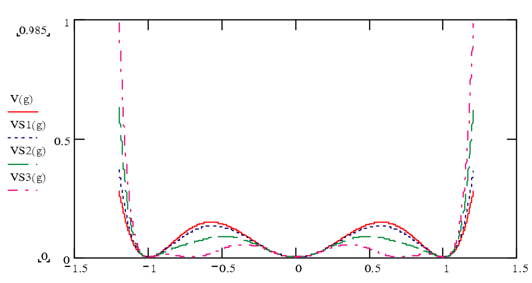

FIG. 2 (color online). We plot the self-dual potential VS for $\theta = 0.5$ and $\tau = 0.5$, 2.0, and 4.0 to show the flattening of the well.

$$F_{sd}(r)/\left(\frac{e^3 v^4}{2\mu^2}\right) \equiv BS(r) = \left[g^2(g^2-1) - \frac{\theta}{4}(g^2-1)\nabla g^2\right],$$

$$-F_{asd}(r)/\left(\frac{e^3 v^4}{2\mu^2}\right) \equiv BA(r) = \left[g^2(g^2-1) + \frac{\theta}{4}(g^2-1)\nabla g^2\right], \quad (71)$$

where as before $B(r)$ is the $\theta = 0$ result. Note that τ does not appear in this relative F profile. The significant facts are: (i) The Chern-Simons-Higgs parameter τ is absent and only the NC parameter θ determines the relative strength of the magnetic fields with or without the NC correction. (ii) The NC effect is not very pronounced for either self-dual or anti-self-dual cases. The universal nature of the NC correction term in the magnetic field that we mentioned before is clearly visible.[3]

In Figs. 4(a)–4(c) , we plot the radial component of the electric field as a function of r given by

$$E_{sd}(r)/\left(\frac{ev^2}{\mu}\right) \equiv ES(r) = gg'[1 - \theta\tau(g^2-1)(3g^2-1)],$$

$$E_{asd}(r)/\left(\frac{ev^2}{\mu}\right) \equiv EA(r) = gg'[1 - \theta\tau(g^4-1)], \quad (72)$$

with $E(r)$ giving the $\theta = 0$ result. The major points to

[3]In this context, we would like to comment that in the work of Lozano *et.al.* in [11], the graphs with values of the parameter $a = (v^2\theta)/(2\kappa^2)$ (where κ is identified with μ in our case), chosen as 2, 1, and 0.5, do not faithfully represent the NC effect. Also one can check that our results do not completely agree quantitatively with Lozano *et al* [11].

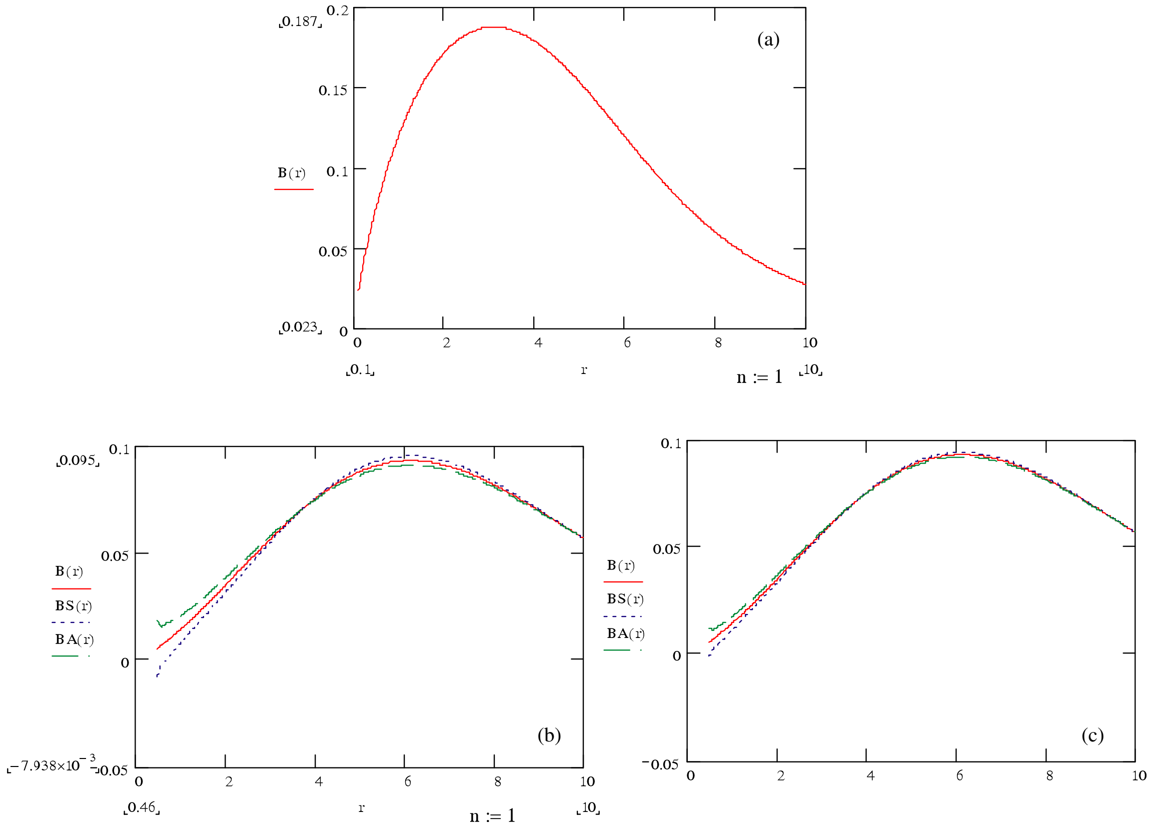

FIG. 3 (color online). We plot the magnetic field B for $\theta = 0$ in (a). In (b) and (c) we plot B, BS (self-dual) and BA (anti-self-dual) profiles for $\theta = 1.0$ and $\theta = 0.5$, respectively. Except for the small r region, the effect of noncommutativity is not very pronounced.

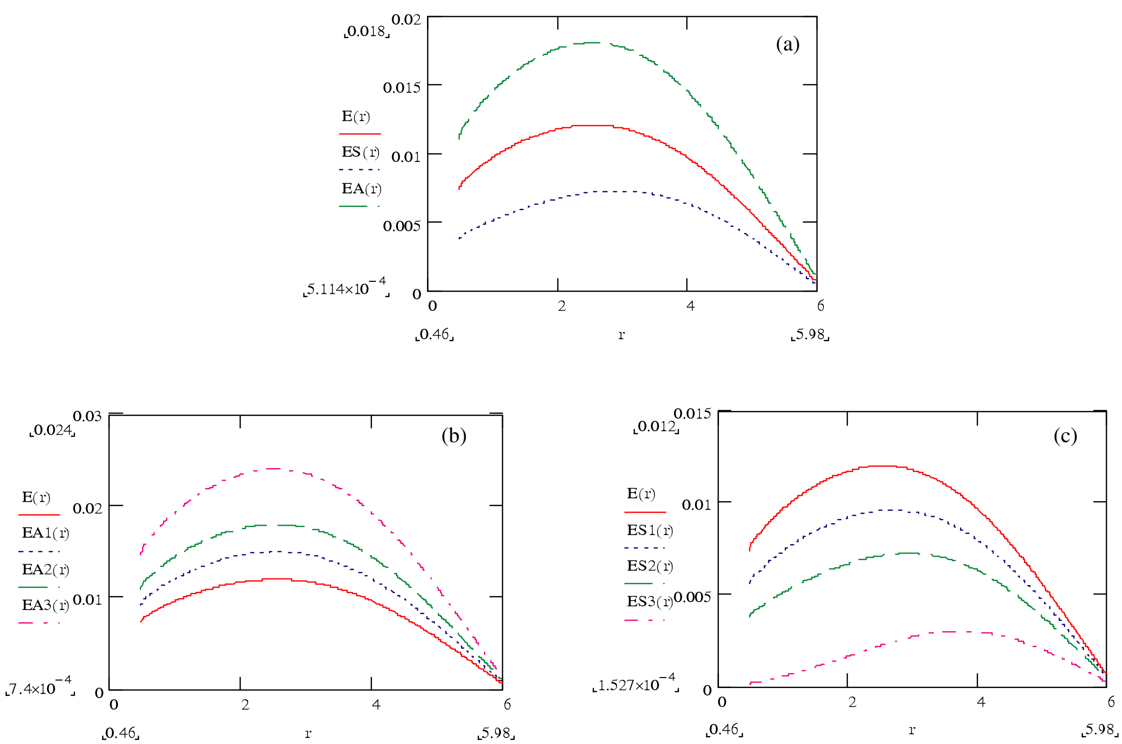

FIG. 4 (color online). In (a) the radial component of the electric field $ES(r)$ for self-dual and $EA(r)$ for anti-self-dual solutions for $\theta = 0.5$ and $\tau = 0.5$ are compared with the electric field $E(r)$ for $\theta = 0$. The effects of the variations of τ are shown in (b) and (c) for self-dual and anti-self-dual solutions, respectively. The values of τ chosen are $\tau = 0.5$, 1.0, and 2.0, for the same $\theta = 0.5$.

note are: (i) Both θ and τ appear in the expressions. (ii) The functional form of the correction terms in the self-dual and anti-self-dual cases are quite distinct.

Lastly in Figs. 5(a)–5(c) we plot the effect of noncommutativity on the energy or, equivalently, the soliton mass, where

$$-W_{sd}(r)/\left(\frac{e^4 v^6}{4\mu^2}\right) \equiv WS(r) = g^2(g^2-1) - \theta\left[\frac{1}{4}(g^2-1)\nabla g^2 + 2\tau g^4(g^2-1)\right],$$
$$-W_{asd}(r)/\left(\frac{e^4 v^6}{4\mu^2}\right) \equiv WA(r) = g^2(g^2-1) + \theta\left[\frac{1}{4}(g^2-1)\nabla g^2 + 2\tau g^4(g^2-1)\right], \tag{73}$$

with $W(r)$ representing the $\theta = 0$ result. There are a couple interesting points to note: (i) Both θ and τ enter the picture. (ii) In the commutative case, the energy is directly proportional to the magnetic field. On the contrary, for nonzero θ, the energy depends as before on the NC magnetic field but there is also the τ-dependent extra contribution

$$WS(r) \equiv BS - 2\theta\tau g^4(g^2-1),$$
$$WA(r) \equiv BA + 2\theta\tau g^4(g^2-1). \tag{74}$$

IX. RESTRICTIONS ON THE HIGGS FIELD

So far we have been able to establish the existence of the BPS soliton unambiguously in the NC Chern-Simons-Higgs model. It is now revealed that the field ϕ has to obey another relation so that the BPS equation becomes compatible with the full variational equation of motion. Obviously this restriction is absent in the commutative case where the BPS equation automatically satisfies the variational equation (as it must). It is worthwhile to point out that we have encountered a similar situation in a previous analysis [13] in the context of the NC $CP(1)$ model.

It would be interesting to find out the reason that induces this consistency condition. Since we have followed the well-established rules of NC extension of a field theory and its subsequent canonical analysis, the above complication is unexpected, at least naively. There might be several reasons for this, the obvious one being that the conventional definitions (i.e., canonical or symmetric) of the EMT are not fully adequate for an arbitrary NC field theory model. In fact, the issue of the correct definition of EMT in a NC theory has been debated [17]. Precisely for this reason we have studied quite thoroughly both forms of the EMT and have demonstrated that they conform to the expected behavior. On the other hand, the very perturbative nature (in θ) of the scheme might be responsible for this departure. This issue requires further study.

Starting with the BPS Eqs. (51) and (52) and the variational equations of motion (9) and (10), we exploit (9), (51), and (52) to reexpress the right-hand side of (10) (the second-order equation for ϕ) in the following form,

$$\frac{1}{2}D_\mu D^\mu \phi + \frac{\delta \hat{V}}{\delta \phi^*} = \frac{\theta e}{4}\Big[\frac{e^3}{2\mu^2} D_i\phi\{(|\phi|^2 - v^2) \times \phi\partial_i\phi^* + 2\partial_i[(|\phi|^2 - v^2) \times |\phi|^2] + 2ie(|\phi|^2 - v^2)|\phi|^2 A_i\} + 2eF^2\phi - ieFA_0D_0\phi\Big]. \tag{75}$$

On the other hand, simplifying the left-hand side of (10), once again by exploiting the BPS Eqs. (51) and (52), one can check that there are no θ-independent terms and the remaining $O(\theta)$ term cancels with the last line of the right-hand side of (75). This shows that the BPS equations and variational (second-order) equations are consistent for the commutative case ($\theta = 0$) and we obtain the cherished compatibility condition

$$D_i\phi\{(|\phi|^2 - v^2)\phi\partial_i\phi^* + 2\partial_i[(|\phi|^2 - v^2)|\phi|^2] + 2ie(|\phi|^2 - v^2)|\phi|^2 A_i\} = 0. \tag{76}$$

Since $D_i\phi$ is nonvanishing in general, (76) leads to an expression for A_i in terms of ϕ. At the same time, one can utilize the BPS Eqs. (51) and (52) to express A_i in terms of ϕ in the following way,

$$A_i = \frac{i}{2e|\phi|^2}(i\epsilon_{ij}\partial_j|\phi|^2 - \phi^*\partial_i\phi + \phi\partial_i\phi^*). \tag{77}$$

Combining the above two forms of A_i we get the equation that has to be satisfied by ϕ. Considering the previously used form of ϕ in (68) we find that (76) is scaled by $\theta e^4 v^4/8\mu^2 \equiv \theta\tau$ and so will be small for small θ and a proper choice of τ. The equation for ϕ slightly restricts the domain of r in $g(r)$ in (68).

X. CONCLUSION

We have studied the noncommutative extension of the relativistic Chern-Simons-Higgs model, in the first nontrivial order in θ. Both Lagrangian and Hamiltonian formulations of the problem have been discussed, with the focus being on the canonical and symmetric forms of the energy-momentum tensor.

In the Hamiltonian scheme, constraint analysis and the induced Dirac brackets have been provided. The space-

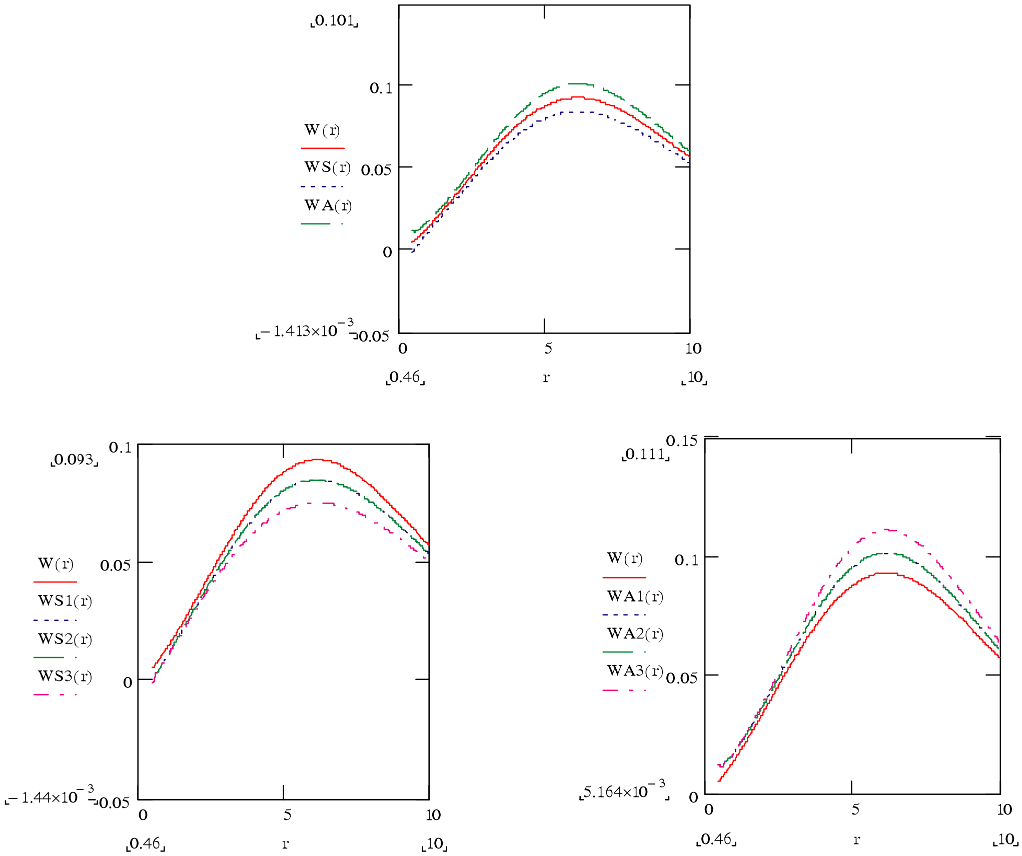

FIG. 5 (color online). In (a) the energy density $W(r)$ for $\theta = 0$ is compared with $WS(r)$ self-dual and $WA(r)$ anti-self-dual solutions for $\theta = 0.5$ and $\tau = 0.5$. The effects of the variations of τ are shown in (b) and (c) for self-dual and anti-self-dual solutions, respectively. The values of τ chosen are $\tau = 0.5$, 1.0, and 2.0, for the same $\theta = 0.5$.

time translation generators and their actions on the fields are discussed in detail.

The BPS soliton solutions are obtained from the energy expression, derived from the symmetric energy-momentum tensor in the Lagrangian framework. It is shown that, in terms of phase space variables, the energy expressions obtained from canonical and symmetric forms of the energy-momentum tensor are identical on the constraint surface. The solitons reduce smoothly to their commutative counterpart in the $\theta = 0$ limit.

We have studied the effects of noncommutativity on the soliton solutions thoroughly and have come up with some interesting observations. Considering the *relative* strength of the noncommutative effects, we have shown that there is a universal character in the noncommutative correction to the magnetic field—it depends *only* on θ. On the other hand, in the cases of all other observables of physical interest, such as the potential profile, soliton mass or the electric field, θ as well as τ (comprised solely of commutative Chern-Simons-Higgs model parameters) appear on similar footings. This phenomenon is a new finding which has come up in the present analysis.

Lastly, we have shown that, unlike the commutative case, the BPS equations do not automatically solve the variational equations of motion and the noncommutativity imposes a further restriction on the Higgs field and the parameters of the theory. A similar situation prevails in the noncommutative $CP(1)$ model [13].

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